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Some Remarks on Dirac's Equation in the Tolman–Bondi Geometry

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The Tolman–Bondi model is studied and integrated via the Newman–Penrose formalism. The results are used to discuss the Dirac equation. It is shown that there exist time regions during which the cosmological background can be considered fixed and the Dirac equation separated and reduced in a standard way to a one-dimensional wave equation.

1. INTRODUCTION

The Tolman–Bondi model (Tolman, 1934; Bondi, 1947) represents a spherically symmetric solution of the Einstein equation for a universe filled with dustlike matter. The interest of this model lies in the fact that the equations of motion, which are exactly integrable, contain solutions which represent an indefinitely expanding universe or a collapsing universe both starting with a big bang (Demianski and Lasota, 1973).

The model has been reconsidered, in an elementary way, in Zecca (1991), by adding a cosmological constant term to the Einstein equations. The solutions resulting from this assumption have been found to be essentially the same of those of the standard case with the adjoint of a harmonic potential to the Kepler-like equation to which the motion can be reduced. However, the presence of the cosmological term leads to incongruities concerning the propagation of light near the collapsing time.

It is the object of this paper to investigate the Dirac equation in the Tolman–Bondi model, which, by the previous consideration, will be considered without the cosmological term.

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The problem is formulated by translating the Dirac equation in Minkowski space into the Dirac equation in the Tolman-Bondi curved space-time via the spinor calculus of the associated Newman-Penrose formalism (Newman and Penrose, 1962). This study requires the preliminary calculation of the spin coefficients and the solution of the corresponding Einstein equation. The so-established Dirac equation has a complex structure owing to the explicit time dependence of the spin coefficients and of the directional derivatives. In a physical situation, namely in the case of the collapsing universe but for time values near half of the collapsing time of the cosmological background and large on a microscopic scale it is shown that the complex time dependence of the Dirac equation disappears and it can be reduced in a well-known standard way to a one-dimensional wave equation.

2. THE TOLMAN-BONDI MODEL

The model consists in a spherically symmetric space-time filled with freely falling dust matter in such a way that in a comoving coordinate system the proper time has the form

$$d\tau^2 = dt^2 - e^{\Gamma}\, dr^2 - Y^2(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{1}$$

with $\Gamma = \Gamma(r, t)$ and $Y = (r, t) > 0$. The Einstein field equations are assumed to be

$$R_{\mu\nu} = -8\pi G(T_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} T^{\lambda}_{\lambda}) = -8\pi G S_{\mu\nu} \tag{2}$$

with $T_{\mu\nu} = \eta U_\mu U_\nu$, $U^t = 1$, $U^i = 0$, $i \neq t$. Explicitly

$$\begin{aligned} S_{rr} &= e^{\Gamma}\frac{\eta}{2}, \qquad & S_{\theta\theta} &= Y^2 \frac{\eta}{2} \\ S_{\phi\phi} &= S_{\theta\theta}\sin^2\theta, \qquad & S_{tt} &= \frac{\eta}{2} \end{aligned} \tag{3}$$

and $S_{\alpha\beta} = 0$ if $\alpha \neq \beta$, where $\eta = \eta(r, t)$ represents the density of the dust matter of negligible pressure. A direct solution of equations (2) together with some physical aspects of the model can be found in Demianski and Lasota (1973). Since the object here is to discuss the Dirac equation in the Tolman-Bondi geometry, an account of the solution of equations (2) in the Newman-Penrose formalism is given. For notations, definitions, and sign conventions, we refer to Chandrasekhar (1983).

The null-tetrad frame is assumed to be

$$
\begin{aligned}
\mathbf{e}^{\mathbf{i}}_{(1)} &= l^i = \frac{1}{\sqrt{2}}(1, e^{-\Gamma/2}, 0, 0)\\
\mathbf{e}^{\mathbf{i}}_{(2)} &= m^i = \frac{1}{\sqrt{2}}(1, -e^{-\Gamma/2}, 0, 0)\\
\mathbf{e}^{\mathbf{i}}_{(3)} &= m^i = \frac{1}{\sqrt{2}Y}(0, 0, 1, i \csc\theta)\\
\mathbf{e}^{\mathbf{i}}_{(4)} &= m^{*i} = (m^i)^*
\end{aligned}
\tag{4}
$$

It satisfies the conventional orthogonality and normalization conditions

$$
\begin{aligned}
\mathbf{l}\cdot\mathbf{m} = \mathbf{l}\cdot\mathbf{m}^* = \mathbf{n}\cdot\mathbf{m} = \mathbf{n}\cdot\mathbf{m}^* = 0\\
\mathbf{l}\cdot\mathbf{n} = \mathbf{m}\cdot\mathbf{m}^* = 1
\end{aligned}
\tag{5}
$$

According to the definition given by Chandrasekhar (1983), the Ricci rotation coefficients, here called spin coefficients, given by

$$\gamma_{(a)(b)(c)} = \tfrac{1}{2}[\lambda_{(a)(b)(c)} + \lambda_{(c)(a)(b)} - \lambda_{(b)(c)(a)}]$$

with

$$\lambda_{(a)(b)(c)} = e_{(b)i,j} = [e^i_{(a)} e^j_{(c)} - e^j_{(a)} e^i_{(c)}]$$

are denoted by the following special symbols with values:

$$
\begin{aligned}
&\kappa = \gamma_{311} = 0 \qquad \nu = \gamma_{242} = 0 \qquad \lambda = \gamma_{244} = 0\\
&\tau = \gamma_{312} = 0 \qquad \pi = \gamma_{241} = 0 \qquad \sigma = \gamma_{313} = 0\\
&\rho = \gamma_{314} = -\frac{1}{\sqrt{2}Y}(\dot{Y} + Y' e^{-\Gamma/2)}\\
&\mu = \gamma_{243} = \frac{1}{\sqrt{2}Y}(\dot{Y} - Y' e^{-\Gamma/2)}\\
&\beta = -\alpha = \tfrac{1}{2}(\gamma_{213} + \gamma_{343}) = \frac{\cot\theta}{2\sqrt{2}Y}\\
&\varepsilon = -\gamma = \tfrac{1}{2}(\gamma_{211} + \gamma_{341}) = \frac{\dot{\Gamma}}{4\sqrt{2}}
\end{aligned}
\tag{6}
$$

(the dot and prime denote partial derivatives with respect to t and r, respectively). According to the Newman–Penrose formalism, the Einstein–Ricci equations corresponding to equations (2) and the Bianchi identities are then obtained in terms of the spin coefficients (6), of the complex scalars

$\Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4$ specifying the Weyl tensor, of suitable expressions of the components of the Ricci tensor and of the directional derivatives $D=l^i\partial_i$, $\Delta=n^i\partial_i$, $\delta=,^i\partial_i$, and $\delta^*=m^{*i}\partial_i$.

In our case, by setting $R_{ik}=e^{\alpha}_{(i)}e^{\beta}_{(k)}R_{\alpha\beta}$, the Ricci tensor is defined through the following scalars having values

$$\begin{aligned}
&\Phi_{00}=-\tfrac{1}{2}R_{11}=2\pi G\eta \\
&\Phi_{11}=-\tfrac{1}{4}(R_{12}+R_{34})=\pi G\eta &\qquad& \Phi_{21}=-\tfrac{1}{2}R_{24}=0 \\
&\Phi_{22}=-\tfrac{1}{2}R_{22}=2\pi G\eta && \Phi_{12}=-\tfrac{1}{2}R_{23}=0 \\
&\Phi_{02}=-\tfrac{1}{2}R_{33}=0 && \Phi_{20}=-\tfrac{1}{2}R_{44}=0 \\
&\Phi_{01}=-\tfrac{1}{2}R_{13}=0 && \Phi_{10}=-\tfrac{1}{2}R_{14}=0 \\
&R=R_{kk}=8\pi G\eta && \Lambda=\tfrac{1}{24}R=\frac{\pi G\eta}{3}
\end{aligned} \tag{7}$$

The Einstein–Ricci equations are tabulated in their general form in Chandrasekhar (1983). Twelve of them come out to be identities in Y or are trivially satisfied in our case, giving

$$\Psi_0=\Psi_1=\Psi_3=\Psi_4=0 \tag{8}$$

The remaining nontrivial equations are

$$D\rho=2\varepsilon\rho+\rho^2+2\pi G\eta \tag{9a}$$

$$(D+\Delta)\gamma=-4\gamma\varepsilon+\Psi_2+\tfrac{2}{3}\pi G\eta \tag{9f}$$

$$D\mu=\rho\mu-2\mu\varepsilon+\Psi_2+\tfrac{2}{3}\pi G\eta \tag{9h}$$

$$(\delta+\delta^*)\alpha=\mu\rho+4\alpha-\Psi_2+\tfrac{4}{3}\pi G\eta \tag{9l}$$

$$\Delta\eta=-\mu^2-2\gamma\mu-2\pi G\eta \tag{9n}$$

$$\Delta\rho=-\rho\mu+2\gamma\rho-\Psi_2-\tfrac{2}{3}\pi G\eta \tag{9q}$$

By taking into account that $D+\Delta=\sqrt{2}\partial_t$, and $D-\Delta=\sqrt{2}e^{-\Gamma/2}\partial_r$, by adding equations (9a) and (9q), subtracting equation (9n) from equation (9h), and then by summing and subtracting the two resulting equations, we get

$$\sqrt{2}(\dot\rho+\dot\mu)=\rho^2-\mu^2 \tag{10}$$

$$\sqrt{2}(\dot\rho-\dot\mu)=(\rho-\mu)^2-2\Psi_2+\tfrac{8}{3}\pi G\eta \tag{11}$$

From (10) and the explicit values (6) of the spin coefficient there follows

$$\dot{\Gamma}=2\frac{\dot{Y}'}{Y'}, \qquad e^{\Gamma}=\frac{Y'^2}{1+2E} \tag{12}$$

$E=E(r)$ is an integration function. Equation (11) gives

$$\Psi_2=\frac{\ddot{Y}}{Y}+\tfrac{4}{3}\pi G\eta \tag{13}$$

With regard to the eight Bianchi identities, they are trivial or automatically satisfied except

$$-D\Psi_2+3\rho\Psi_2-2\pi G\Delta\eta-\tfrac{2}{3}\pi GD\eta+2\pi G\eta(\rho-\mu+4\gamma)=0 \tag{14b}$$

$$-\Delta\Psi_2-3\mu\Psi_2-2\pi GD\eta-\tfrac{2}{3}\pi G\Delta\eta+2\pi G\eta(\rho-\mu+4\gamma)=0 \tag{14g}$$

$$-2\pi G(D+\Delta)\eta=4\pi G\eta(-\rho+\mu-2\gamma) \tag{14i}$$

With the spin coefficient values (6) and the result (12), equation (14i) implies

$$\frac{\partial}{\partial t}(\eta Y^2 Y')=0 \tag{15}$$

We choose the integration function so that

$$m'(r)=4\pi G\eta Y^2 Y', \qquad m(r)=4\pi G\int_0^r \eta Y^2 Y'\,dr \tag{16}$$

By subtracting equation (14g) from equation (14b), using the definition of the directional derivatives and the results (6) and (13), we obtain $(\ddot{Y}Y)'=-m'(r)$ or

$$\ddot{Y}Y=-m(r) \tag{17}$$

with a suitable choice of the integration function. Hence, from (13) and (17)

$$\Psi_2=-\frac{m(r)}{Y^3}+\tfrac{4}{3}\pi G\eta \tag{18}$$

Finally, from equation (91), by taking into account that $\delta+\delta^*=(\sqrt{2}/Y)\,\partial_\theta$ and by using (18), one gets the well-known result

$$\frac{\dot{Y}^2}{2}-\frac{m}{Y}=E \tag{19}$$

One can check that all the remaining equations are identities in Y when Y satisfies equation (19) and Ψ_2 has the expression in (18). From the physical point of view, equation (19) is interpreted as governing a Newtonian-like equation of isotropic gravitating dust matter. This leads to the interpretation of the conserved quantity in (16) as the effective mass of a sphere of radius Y. Equation (19) can be explicitly integrated. The parametric form of the solution is given by (Demianski and Lasota, 1973)

$$\begin{aligned} Y &= G\frac{m(r)}{2E(r)}(\cosh\eta - 1) \qquad (\eta > 0) \\ t - t_0(r) &= G\frac{m(r)}{[2E(r)]^{3/2}}(\sinh\eta - \eta) \end{aligned} \tag{20}$$

for $E > 0$, while

$$\begin{aligned} Y &= G\frac{m(r)}{-2E(r)}(\cos\eta - 1) \qquad (0 \leq \eta \leq 2\pi) \\ t - t_0(r) &= G\frac{m(r)}{[-2E(r)]^{3/2}}(\eta - \sin\eta) \end{aligned} \tag{21}$$

for $E < 0$. If we take $t_0(r) = 0$, the case $E > 0$ represents an indefinitely expanding universe starting with a big bang at time $t = 0$, while the case $E < 0$ can be interpreted as representing an initially expanding universe starting with a big bang at time $t = 0$ and such that the dust matter at distance r has been collapsed at time $t_c(r) = 2\pi Gm(r)/[-2E(r)]^{3/2}$.

3. SOME REMARKS ON THE DIRAC EQUATION

The Dirac equation in spinorial form can be expressed in the Newman–Penrose formalism by replacing the ordinary derivatives and Pauli matrices by the covariant derivatives and a generalization of the Pauli matrices expressed in terms of the components of the vectors of the tetrad frame (Chandrasekhar, 1983, Section 103).

The equations so obtained can be written in explicit form in terms of the directional derivatives and the spin coefficients. In the Tolman–Bondi geometry they read

$$\begin{aligned} (D + \varepsilon - \rho)F_1 + (\delta^* - \alpha)F_2 &= i\mu_* G_1 \\ (\Delta + \mu + \varepsilon)F_2 + (\delta - \alpha)F_1 &= i\mu_* G_2 \\ (D + \varepsilon - \rho)G_2 - (\delta - \alpha)G_1 &= i\mu_* F_2 \\ (\Delta + \mu + \varepsilon)G_1 - (\delta^* - \alpha)G_2 &= i\mu_* F_1 \end{aligned} \tag{22}$$

$\mu_*\sqrt{2}$ is the mass of the particle, F_1, F_2, G_1, and G_2 are the four components of the wave function, which depends on r, θ, ϕ, and t, and the spin coefficients have the explicit form (6).

The solution of equations (22) in their complete form seems to be quite complicated since both the spin coefficients and the directional derivatives have an explicit time dependence, $Y(r, t)$ being the solution of equation (19). So we are led to look for physical situations or approximations in which equations (22) can be separated. From the symmetry of the metric (1) it is evident that $(\partial\phi)^\mu$ is a Killing vector. The covariant form of the Killing equation for $(\partial t)^\mu$ is $\nabla_a(\partial t)_b + \nabla_b(\partial t)_a = 0$. In the comoving coordinate system that has been employed in equation (1) one has $(\partial t)^\mu \equiv (1, 0, 0, 0) \equiv (\partial t)_\mu$. Furthermore,

$$\nabla_\mu(\partial t)^\nu = \partial_\mu(\partial t)^\nu + \Gamma^\nu_{\mu\alpha}(\partial t)^\alpha = \Gamma^\nu_{\mu t}$$

so that

$$\Gamma^\nu_{\mu t} + \Gamma^\mu_{\nu t} = 0 \tag{23}$$

Equation (23) is not a covariant equation, since it is not identically satisfied. Indeed the nonzero Γ^b_{at} can be directly obtained from the metric (1) to give

$$\Gamma^\phi_{\phi t} = \Gamma^\theta_{\theta t} = \frac{\dot{Y}}{Y}, \qquad \Gamma^r_{rt} = 2\frac{\dot{Y}'}{Y'} \tag{24}$$

We notice, however, that the Dirac equation describes physical processes taking place in time intervals large on a microscopical scale but small on a macroscopic and hence on a cosmological scale. Since $\dot{Y}(r, \frac{1}{2}t_c) = 0$ in the case of equation (21), the cosmological background $Y(r, t)$ can be considered fixed for time intervals $|t - \frac{1}{2}t_c| \ll t_c$, but large on a microscopic scale. In these intervals the Dirac equation can be solved by setting $\dot{Y}(r, t) \simeq 0$, $Y(r, t) \simeq Y(r, \frac{1}{2}t_c) \equiv Gm/E$, and hence by taking the spin coefficients and the tetrad frame vectors as independent of the time t.

Under these assumptions $(\partial t)^a$ and $(\partial\phi)^b$ can be considered as commuting Killing vectors and the standard dependence $\exp[i(\sigma t + m\phi)]$ on ϕ and t of the wave function together with the approximations

$$\varepsilon - \rho \cong \frac{(E + \frac{1}{2})^{1/2}}{Y} \tag{25}$$

$$\mu - \gamma \cong -\frac{(E + \frac{1}{2})^{1/2}}{Y} \tag{26}$$

can be assumed in solving equation (22). Then the Dirac equation (22) can be further simplified by applying the well-known method of separation exposed by Chandrasekhar (1983, Chapter 10). From (4), (6), (12), and (26) and the definitions of the directional derivatives, equations (22) become

$$\begin{aligned} D_0(YF_1)+2^{-1/2}L_{1/2}F_2&=i\mu_* YG_1\\ D_0^+(YF_2)-2^{-1/2}L_{1/2}^+F_1&=-i\mu_* YG_2\\ D_0(YG_2)-2^{-1/2}L_{1/2}^+G_1&=i\mu_* YF_2\\ D_0^+(YG_1)+2^{-1/2}L_{1/2}G_2&=-i\mu_* YF_1 \end{aligned} \tag{27}$$

where we have set

$$\begin{aligned} D_0&=\frac{1}{\sqrt{2}}(e^{-\Gamma/2}\partial_r+i\sigma), \qquad L_{1/2}=\partial_\theta+m\csc\theta+\tfrac{1}{2}\cot\theta\\ D_0^+&=\frac{1}{\sqrt{2}}(e^{-\Gamma/2}\partial_r-i\sigma), \qquad L_{1/2}^+=\partial_\theta-m\csc\theta+\tfrac{1}{2}\cot\theta \end{aligned} \tag{28}$$

and the relation

$$D_0(YF)=YD_0F+F(E+\tfrac{1}{2})^{1/2}$$

together with

$$D_0^+(YF)=YD_0^+F+F(E+\tfrac{1}{2})^{1/2}$$

has been used. The r and θ dependences can be separated in (27) by $YF_1=R_{-1/2}(r)S_{-1/2}(\theta)$, $YF_2=R_{1/2}(r)S_{1/2}(\theta)$, $YG_1=R_{1/2}(r)S_{-1/2}(\theta)$, and $YG_2=R_{-1/2}(r)S_{1/2}(\theta)$ to obtain the Teukolsky-like equations

$$\begin{aligned} YD_0R_{-1/2}&=(\chi+i\mu_* Y)R_{1/2}\\ YD_0^+R_{1/2}&=(\chi-i\mu_* Y)R_{-1/2} \end{aligned} \tag{29}$$

together with standard equations for $S_{\pm1/2}$ which determine the value of the constant χ (see, for instance, Chandrasekhar, 1983).

By successively setting

$$\begin{aligned} r_*&=\sqrt{2}\int_0^r \frac{Y'}{(1+2E)^{1/2}}\,dr\\ \chi\pm i\mu_* Y&=(\chi^2+\mu_*^2Y^2)^{1/2}\exp\left(\pm i\,\mathrm{tg}^{-1}\frac{\mu_* Y}{\chi}\right)\\ R_{\pm1/2}&=\Psi_{\pm1/2}\exp\left[\mp\tfrac{1}{2}i\,\mathrm{tg}^{-1}\left(\frac{\mu_* Y}{\chi}\right)\right] \end{aligned} \tag{30}$$

one gets, from equation (29),

$$\frac{d\Psi_{\pm 1/2}}{dr_*} \mp i\sigma\left(1+\frac{(E+1/2)^{1/2}\chi\mu_*}{\chi^2+\mu_*^2 Y^2}\right)\Psi_{\pm 1/2} = \frac{(\chi^2+\mu_*^2Y^2)^{1/2}}{Y}\Psi_{\mp 1/2} \tag{31}$$

Moreover, by setting

$$\hat{r}_* = r_* + \operatorname{tg}^{-1}\left(\frac{\mu_* Y}{\chi}\right) \tag{32}$$

one gets from equation (31) that the functions $\Psi_{\pm 1/2}$ satisfy the equations

$$\begin{gathered}\left(\frac{d}{d\hat{r}_*} \mp i\sigma\right)\Psi_{\pm 1/2} = W\Psi_{\mp 1/2} \\ W = \frac{\sqrt{2}(\chi^2+\mu_*^2Y^2)^{3/2}}{Y[(\chi^2+\mu_*^2Y^2)\sqrt{2}+\chi\mu_*(1+2E)^{1/2}]}\end{gathered} \tag{33}$$

Taking further

$$Z_{\pm 1/2} = \Psi_{1/2} \pm \Psi_{-1/2} \tag{34}$$

one obtains

$$\left(\frac{d^2}{d\hat{r}_*^2} + \sigma^2\right)Z_\pm = V_\pm Z_\pm \tag{35}$$

where

$$\begin{aligned} V_\pm &= W^2 \pm \frac{dW}{d\hat{r}_*} \\ &= \frac{2(\chi^2+\mu_*^2Y^2)^3}{Y^2[\sqrt{2}(\chi^2+\mu_*^2Y^2)+\chi\mu_*(1+2E)^{1/2}]^2} \\ &\quad \pm \frac{2(\chi^2+\mu_*^2Y^2)}{Y^2[\sqrt{2}(\chi^2+\mu_*^2Y^2)+\chi\mu_*(1+2E)^{1/2}]^3} \\ &\quad \times \left\{\frac{\partial Y}{\partial r_*}(\chi^2+\mu_*^2Y^2)^{1/2}[\sqrt{2}(\chi^2+\mu_*^2Y^2)+\chi\mu_*(1+2E)^{1/2}](2\mu_*^2Y^2-\chi^2)\right. \\ &\quad \left. + Y(\chi^2+\mu_*^2Y^2)^{3/2}\left[2\mu_*^2Y(1+2E)^{1/2} + \frac{dE/dr_*}{(1+2E)^{1/2}}\chi\mu_*\right]\right\} \end{aligned} \tag{36}$$

With regard to the case $E>0$, even if the condition $\dot{Y}/Y=0$ is rapidly approximated for increasing t, Y can no longer be considered as time independent, since, from (20), $Y \simeq t(2E)^{1/2}$ for large t.

The study of this last case as well as the case for times near the collapsing time when $E<0$ would be of interest to throw more light on the exact solution of the Dirac equation (22).

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